# Inverse scattering for the plasma wave equation starting with large-t data



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# Inverse scattering for the plasma wave equation starting with large-*t* data

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**Abstract.** The inverse scattering problem for the plasma wave equation is approached from the viewpoint of scattering theory for the reduced wave equation. It is shown that the scattered wave for the plasma wave equation has the large-$t$ asymptotics which are expressed by the scattering amplitude for the Schrödinger operator associated with the plasma wave equation.

## 1. Introduction

The inverse scattering problem for low-density plasmas such as the electron density in Earth's atmosphere has been studied by many authors including Jordan and Ahn (1979), Morawetz (1981), Morawetz and Kriegsmann (1983), DeFacio and Rose (1985), Rose *et al* (1985). All of these works have used either the time-domain or the $L^2$-kernel properties to establish the results. The role played by spectral theory is vanishingly small in the above works. Lax and Phillips (1967) and Wilcox (1975) have developed the spectral theory of wave equations. Wilcox's approach will be shown to give a clear and simple treatment of the inverse scattering problem of the three-dimensional plasma equation

$$(\Delta-\partial_t^2-V(x))u(t,x)=0 \tag{1.1}$$

where $t\in\mathbb{R}$, $x\in\mathbb{R}^3$, $\Delta$ is the Laplacian and $\partial_t^2=\partial^2/\partial t^2$. The relation of the reduced or Helmholtz equation to time-domain approach will also be clarified. One of the consequences of the main theorem (theorem 2.2) is that the large-$t$ asymptotics $u_\infty^{\mathrm{sc}}$ are the same as the large-$x$ asymptotics in the sense that $u_\infty^{\mathrm{sc}}$ is expressed by the scattering amplitude $F(k,\omega,\omega')$ for the Schrödinger operator associated with the plasma wave equation. It is hoped that these methods may persuade some applied mathematicians and theoretical physisists to adopt them in formulating their work. It is clear that the spectral approach probably will not solve their calculation problems.

In § 2 we shall present the asymptotic formula for the scattering wavefunction $u^{\mathrm{sc}}(t,x)$ which is defined by

$$u^{\mathrm{sc}}(t,x)=u(t,x)-u_0(t,x) \tag{1.2}$$

where $u(t,x)$ is the solution of the equation (1.1) with the initial condition and $u_0(t,x)$ is the solution of the wave equation

$$[\Delta-\partial_t^2]u_0(t,x)=0 \tag{1.3}$$

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with the same initial condition. A proof of the asymptotic formula will be given in § 3 where we shall essentially follow the arguments of Wilcox (1975, 1983). In § 4 the inverse scattering problem for the plasma wave equation will be discussed. From the asymptotic formula we recover the scattering amplitude associated with the Schrödinger operator

$$H = -\Delta + V(x) \tag{1.4}$$

so that we can reduce the inverse scattering problem for the plasma wave equation to the inverse scattering problem for the Schrödinger operators.

## 2. Asymptotic formula for the plasma wave equation

Let us consider the plasma wave equation (1.1), where $V(x)$ satisfies the following assumption.

*Assumption 2.1.* $V(x)$, $x \in \mathbb{R}^3$, is a non-negative, bounded function with compact support, i.e. $V(x)$ is a measurable function defined on $\mathbb{R}^3$ satisfying

$$0 \leqslant V(x) \leqslant C_0 \qquad x \in \mathbb{R}^3 \tag{2.1}$$

and

$$V(x) = 0 \qquad |x| \geqslant R_0 \tag{2.2}$$

with positive $C_0$ and $R_0$.

Let $H$ be the self-adjoint realisation of the differential operator $-\Delta + V(x)$ in $L^2(\mathbb{R}^3)$, i.e.

$$Hu = -\Delta u + V(x)u \qquad D(H) = H^2(\mathbb{R}^3) \tag{2.3}$$

where $D(H)$ means the domain of $H$ and $H^2(\mathbb{R}^3)$ is the Sobolev space of the second order. Notice that the non-negativeness of the potential $V(x)$ would exclude the existence of the eigenvalues of $H$ which were discussed by Newton (1985).

Let $s(\tau)$ be a real-valued $C^2$ function on $\mathbb{R}$ such that

$$\operatorname{supp}(s) \subset [a, b] \tag{2.4}$$

with $-\infty < a < b < \infty$. Let $\theta \in S^2$ and set

$$u_0(t, x) = u_0(t, x, \theta) = s(x \cdot \theta - t) \tag{2.5}$$

where $x \cdot \theta$ denotes the inner product in $\mathbb{R}^3$. Then $u_0(x, t)$ satisfies the (free) wave equation (1.3). Since we have

$$\begin{gathered} V(x)u_0(t, x, \theta) = 0 \\ ((t, x) \in (-\infty, t_0) \times \mathbb{R}^3 \text{ or } (t_1, \infty) \times \mathbb{R}^3) \end{gathered} \tag{2.6}$$

with

$$t_0 = -b - R_0 \qquad \text{and} \qquad t_1 = -a + R_0 \tag{2.7}$$

the positive constant $R_0$ being given in assumption 2.1. Thus $u_0(t, x, \theta)$ satisfies the plasma wave equation (1.1) for $t \leqslant t_0$ and $t \geqslant t_1$. Let $u(t, x, \theta)$ be the solution of the plasma wave equation (1.1) with the initial condition

$$u(t_0, x, \theta) = u_0(t_0, x, \theta) \qquad \partial_t u(t_0, x, \theta) = \partial_t u_0(t_0, x, \theta). \tag{2.8}$$

Let us define the scattered wave $u^{sc}(t, x, \theta)$ by

$$u^{sc}(t, x, \theta) = u(t, x, \theta) - u_0(t, x, \theta). \tag{2.9}$$

Then $u^{sc}(t, x, \theta)$ satisfies the equation

$$[\Delta - \partial_t^2 - V(x)]u^{sc} = V(x) u_0(t, x, \theta) \tag{2.10}$$

with

$$u^{sc}(t, x, \theta) = \partial_t u^{sc}(t, x, \theta) = 0 \qquad t \leqslant t_0, \, x \in \mathbb{R}^3. \tag{2.11}$$

In order to study the asymptotic behaviour of $u^{sc}(t, x, \theta)$ as $t \to \infty$ we need some preparations. Let $H$ be the Schrödinger operator defined as above. The scattering amplitude $F(k, \omega, \omega')$ is defined by

$$F(k, \omega, \omega') = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \varphi(y, -k\omega) V(y) \, e^{iky\cdot\omega'} \, dy \tag{2.12}$$

where $k > 0$, $\omega, \omega' \in S^2$, and $\varphi(x, \xi)(x, \xi \in \mathbb{R}^3)$ is a (unique) solution of the Lippmann-Schwinger equation

$$\varphi(x, \xi) = e^{ix\cdot\xi} - \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\exp(i|\xi||x-y|)}{|x-y|} V(y) \varphi(y, \xi) \, dy \tag{2.13}$$

(see, e.g., Ikebe (1960), Amrein *et al* (1977)). Let us now introduce the far-field space solution

$$u_\infty^{sc}(t, x) = u_\infty^{sc}(t, x, \theta, s) = |x|^{-1} K(|x| - t, \omega_x, \theta, s) \tag{2.14}$$

with $\omega_x = x/|x|$ and

$$K(\nu, \omega, \theta, s) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{i\nu\rho} \hat{s}(\rho) F(\rho, \omega, \theta) \, d\rho \tag{2.15}$$

where $\hat{s}$ is the one-dimensional Fourier transform of $s(\tau)$, i.e.

$$\hat{s}(\rho) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{-i\rho\tau} s(\tau) \, d\tau \tag{2.16}$$

and for $\rho < 0$ $F(\rho, \omega, \theta)$ is defined by

$$F(\rho, \omega, \theta) = F(-\rho, -\omega, -\theta). \tag{2.17}$$

*Theorem 2.2.* Assume assumption 2.1. Let $u^{sc}(t, x, \theta, s)$ and $u_\infty^{sc}(t, x, \theta, s)$ be as above. Then we have

$$u^{sc}(t, x, \theta, s) \approx u_\infty^{sc}(t, x, \theta, s) \tag{2.18}$$

as $t \to \infty$ in the sense that

$$\lim_{t\to\infty} \| u^{sc}(t, \cdot, \theta, s) - u_\infty^{sc}(t, \cdot, \theta, s) \|_{L^2(\mathbb{R}^3)} = 0 \tag{2.19}$$

where $\| \ \|_{L^2(\mathbb{R}^3)}$ denotes the norm of $L^2(\mathbb{R}^3)$.

The proof of theorem 2.2 will be given in § 3. Wilcox (1975) gave a similar formula for the wave equation (1.3) in an exterior domain. In Wilcox (1983) he treated the case of the acoustic wave equation

$$\partial_t^2 u - c(x)^2\rho(x)\nabla\cdot(\rho(x)^{-1}\nabla u) = 0 \tag{2.20}$$

with positive functions $c(x)$ and $\rho(x)$ which have compact support.

## 3. Proof of theorem 2.2

Let us start with the inhomogeneous wave equation (2.10) for the scattered wave $u^{sc}(t, x, \theta)$ with the initial condition (2.11), i.e.

$$\begin{aligned} &\partial_t^2 u^{sc} + Hu^{sc} = -V(x)u_0(t, x, \theta) \equiv Q(t, x, \theta) \\ &u^{sc}(t, x) = \partial_t u^{sc}(t, x) = 0 \qquad t \leqslant t_0;\ x \in \mathbb{R}^3 \end{aligned} \tag{3.1}$$

where $H$ is the Schrödinger operator given by (2.3). Let us note that the operators $H^{1/2}$ and $H^{-1/2}$ are well defined because $H \geqslant 0$ by assumption 2.1 and that

$$\int_{t_0}^{t_1} \exp(\mathrm{i}\tau H^{1/2})Q(\tau, \cdot, \theta, s)\,\mathrm{d}\tau = \int_{-\infty}^{\infty} \exp(\mathrm{i}\tau H^{1/2})Q(\tau, \cdot, \theta, s)\,\mathrm{d}\tau \tag{3.2}$$

because of (2.6). Then by the Duhamel integral we have for $t > t_1$

$$\begin{aligned} u^{sc}(t, x, \theta) &= \mathrm{Re}\left(\mathrm{i}H^{-1/2}\int_{t_0}^{t_1} \exp[-\mathrm{i}(t-\tau)H^{1/2}]Q(\tau, \cdot, \theta, s)\,\mathrm{d}\tau\right) \\ &= \mathrm{Re}[\exp(-\mathrm{i}tH^{1/2})h] \end{aligned} \tag{3.3}$$

with

$$h = h(x, \theta, s) = \mathrm{i}H^{-1/2}\int_{-\infty}^{\infty} \exp(\mathrm{i}\tau H^{1/2})Q(\tau, \cdot, \theta, s)\,\mathrm{d}\tau \tag{3.4}$$

(cf Wilcox 1975).

We need some results from the time-dependent scattering theory for the Schrödinger operators. Let us denote by $H_0$ the self-adjoint realisation of $-\Delta$, i.e.

$$H_0 = -\Delta u \qquad D(H_0) = H^2(\mathbb{R}^3). \tag{3.5}$$

Then it is well known that there exist the wave operators $W_\pm$ given by

$$W_\pm = \operatorname*{s-lim}_{t\to\pm\infty} \exp(\mathrm{i}tH)\exp(-\mathrm{i}tH_0) \tag{3.6}$$

where s-lim means the strong limit in $L^2(\mathbb{R}^3)$, and that the operators $W_\pm$ are complete (e.g., Ikebe 1960, Amrein *et al* 1977). Further, it follows from the invariance principle (e.g., Kato 1976, ch 10) that we have

$$W_\pm = \operatorname*{s-lim}_{t\to\pm\infty} \exp(\mathrm{i}tH^{1/2})\exp(-\mathrm{i}tH_0^{1/2}). \tag{3.7}$$

Therefore the adjoint operators $U_\pm = W_\pm^*$ of $W_\pm$ are given by

$$U_\pm = \operatorname*{s-lim}_{t\to\pm\infty} \exp(\mathrm{i}tH_0^{1/2})\exp(-\mathrm{i}tH^{1/2}). \tag{3.8}$$

Let us introduce the complex-valued scattered wave $v^{sc}$ by

$$v^{sc}=v^{sc}(t,x,\theta,s)=\exp(-\mathrm{i}tH^{1/2})h \tag{3.9}$$

with $h$ given by (3.4). Then, noting that

$$v^{sc}=\exp(-\mathrm{i}tH_0^{1/2})\exp(\mathrm{i}tH_0^{1/2})\exp(-\mathrm{i}tH^{1/2})h \tag{3.10}$$

we see that

$$v^{sc}(\cdot,t,\theta,s)\approx\exp(-\mathrm{i}tH_0^{1/2})U_+h(\cdot,\theta,s) \tag{3.11}$$

as $t\to\infty$ in the sense that

$$\lim_{t\to\infty}\|v^{sc}(t,x,\theta,s)-\exp(-\mathrm{i}tH_0^{1/2})U_+g(\cdot,\theta,s)\|_{L^2(\mathbb{R}^3)}=0. \tag{3.12}$$

Let $g\in L^2(\mathbb{R}^3)$ and set

$$w=\exp(-\mathrm{i}tH_0^{1/2})g. \tag{3.13}$$

As is shown by Wilcox (1975), $w$ satisfies the following asymptotic formula:

$$w(x,t)\approx|x|^{-1}G_0(|x|-t,\omega_x,g)\qquad t\to\infty \tag{3.14}$$

where $\omega_x=x/|x|$,

$$G_0(\nu,\omega,g)=\frac{1}{(2\pi)^{3/2}}\int_0^\infty\exp(\mathrm{i}\nu\rho)(\mathscr{F}_0g)(\xi)|_{\xi=\rho\omega}(-\mathrm{i}\rho)\,\mathrm{d}\rho \tag{3.15}$$

and $(\mathscr{F}_0g)(\xi)$ is the usual Fourier transform of $g$, i.e.

$$(\mathscr{F}_0g)(\xi)=\operatorname*{s-lim}_{R\to\infty}\frac{1}{(2\pi)^{1/2}}\int_{|x|<R}\exp(-\mathrm{i}\xi\cdot y)g(y)\,\mathrm{d}y \tag{3.16}$$

in $L^2(\mathbb{R}^3_\xi)$ with $\rho>0$ and $\omega\in S^2$. Setting $g=U_+h$ in (3.14), we get

$$\exp(-\mathrm{i}tH_0^{1/2})U_+h\approx|x|^{-1}G_0(|x|-t,\omega_x,U_+h) \tag{3.17}$$

as $t\to\infty$. Let us define $G(\nu,\omega,\theta,s)$ by

$$G(\nu,\omega,\theta,s)=G_0(\nu,\omega,U_+h(\cdot,\theta,s)) \tag{3.18}$$

and set

$$v^{sc}_\infty(t,x,\theta,s)=|x|^{-1}G(|x|-t,\omega_x,\theta,s). \tag{3.19}$$

Then it follows from (3.11), (3.17)-(3.19) that the asymptotic formula

$$v^{sc}(t,x,\theta,s)\approx v^{sc}_\infty(t,x,\theta,s) \tag{3.20}$$

as $t\to\infty$, i.e.

$$\lim_{t\to\infty}\|v^{sc}(t,\cdot,\theta,s)-v^{sc}_\infty(t,\cdot,\theta,s)\|_{L^2(\mathbb{R}^3)}=0. \tag{3.21}$$

Let us study the asymptotic wavefunction $G(\nu,\omega,\theta,s)$ for the complex-valued scattered wave $v^{sc}(t,x,\theta,s)$. It follows from the stationary scattering theory that

$$U_+h=W_\pm^*h=\mathscr{F}_0^*\mathscr{F}_+h \tag{3.22}$$

where $\mathscr{F}_0^*$ is the adjoint of the Foueier transform $\mathscr{F}_0$ and $\mathscr{F}_+$ is the generalised Fourier transform associated with the Schrödinger operator $H=-\Delta+V(x)$ given by

$$(\mathscr{F}_+h)(\xi)=\frac{1}{(2\pi)^{3/2}}\operatorname*{s-lim}_{R\to\infty}\int_{|x|<R}\varphi(y,-\xi)h(y)\,\mathrm{d}y \tag{3.23}$$

in $L_2(\mathbb{R}^3_\xi)$. The operator $\mathscr{F}_+$ is known to be a unitary operator from $L^2_x(\mathbb{R}^3) = L^2(\mathbb{R}^3, \mathrm{d}x)$ onto $L^2_\xi(\mathbb{R}^3) = L^2(\mathbb{R}^3, \mathrm{d}\xi)$ such that

$$\mathscr{F}_+^* C_{\sqrt{I}} \mathscr{F}_+ = E(I) \tag{3.24}$$

where $\mathscr{F}_+^*$, the adjoint of $\mathscr{F}_+$, is a unitary operator from $L^2_\xi(\mathbb{R}^3)$ onto $L^2_x(\mathbb{R}^3)$, $I$ is a Borel set in $(0, \infty)$, $C_{\sqrt{I}}(\xi)$ is the characteristic function defined by

$$C_{\sqrt{I}}(\xi) = \begin{cases} 1 & \text{if } |\xi|^2 \in I \\ 0 & \text{otherwise} \end{cases} \tag{3.25}$$

and $E(\cdot)$ is the spectral measure associated with the Schrödinger operator $H$. It is also well known that assumption 2.1 is strong enough to guarantee the asymptotic completeness, i.e. the range of $E((0, \infty))$ is equal to both of the incoming and outgoing spaces. Since $H \geqslant 0$ in our case, the generalised Fourier transform $\mathscr{F}_+$ gives the spectral representation of $H$. It also follows the absolute continuity of $H$ (see, e.g., Ikebe 1960, Simon 1971, Amrein *et al* 1977). From (3.15) and (3.22) we obtain

$$G(\nu, \omega, \theta, s) = G_0(\nu, \omega, \mathscr{F}_0^* \mathscr{F}_+ h) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\nu\rho} (\mathscr{F}_+ h)(\xi)|_{\xi = \rho\omega} (-\mathrm{i}\rho)\, \mathrm{d}\rho \tag{3.26}$$

where we should notice the relation $\mathscr{F}_0^* \mathscr{F}_0 = \text{identity}$. Further, recalling the definition of $h(x, \theta, s)$ and $\mathscr{F}_+$ ((3.3) and (3.23)) and using the well known relation

$$\mathscr{F}_+ \Phi(H) = \Phi(|\xi|^2) \mathscr{F}_+ \tag{3.27}$$

with a Borel measurable function $\Phi(k)$ defined on $(0, \infty)$, we obtain

$$\begin{aligned}
(\mathscr{F}_+ h)(\xi) &= \mathscr{F}_+\left(\mathrm{i}H^{-1/2} \int_{-\infty}^{\infty} \exp(\mathrm{i}\tau H^{1/2}) Q(\cdot, \theta, s)\, \mathrm{d}\tau\right) \\
&= \int_{-\infty}^{\infty} \mathrm{i}|\xi|^{-1} \exp(\mathrm{i}\tau|\xi|) \mathscr{F}_+ Q(\cdot, \theta, s)\, \mathrm{d}\tau \\
&= \frac{-\mathrm{i}}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} |\xi|^{-1} \exp(\mathrm{i}\tau|\xi|) \left( \int_{\mathbb{R}^3} \varphi(y, -\xi) V(y) s(y \cdot \theta - \tau)\, \mathrm{d}y \right) \mathrm{d}\tau \\
&= -\mathrm{i}\hat{s}(|\xi|)(2\pi|\xi|)^{-1} \int_{\mathbb{R}^3} \varphi(y, -\xi) V(y) \exp(\mathrm{i}|\xi| y \cdot \theta)\, \mathrm{d}y
\end{aligned} \tag{3.28}$$

with the one-dimensional Fourier transform $\hat{s}(|\xi|)$ defined by (2.16). Thus, setting $\xi = \rho\omega$ in (3.28), we arrive at

$$\begin{aligned}
G(\nu, \omega, \theta, s) &= -\frac{1}{(2\pi)^{3/2}} \int_0^{\infty} \mathrm{e}^{\mathrm{i}\nu\rho} \left( \int_{\mathbb{R}^3} \varphi(y, -\rho\omega) V(y) \exp(\mathrm{i}\rho y \cdot \theta)\, \mathrm{d}y \right) \hat{s}(\rho)\, \mathrm{d}\rho \\
&= \left(\frac{2}{\pi}\right)^{1/2} \int_0^{\infty} \mathrm{e}^{\mathrm{i}\nu\rho} \hat{s}(\rho) F(\rho, \omega, \theta)\, \mathrm{d}\rho
\end{aligned} \tag{3.29}$$

with the scattering amplitude $F(\rho, \omega, \theta)$ given by (2.12).

Since we have found the asymptotic wavefunction $G(\nu, \omega, \theta, s)$ for the complex-valued scattered wave $v^{\mathrm{sc}}(t, x, \theta, s)$, it is now easy to find the asymptotic wavefunction $K(\nu, \omega, \theta, s)$ for the real-valued scattered wave $u^{\mathrm{sc}}(t, x, \theta, s)$. Obviously $K(\nu, \omega, \theta, s)$ should be defined as

$$K(\nu, \omega, \theta, s) = \mathrm{Re}(G(\nu, \omega, \theta, s)). \tag{3.30}$$

Thus it follows from (3.29) that

$$K(\nu,\omega,\theta,s)=\tfrac{1}{2}(G(\nu,\omega,\theta,s)+\overline{G(\nu,\omega,\theta,s)})$$
$$=\frac{1}{(2\pi)^{1/2}}\int_0^\infty (\mathrm{e}^{\mathrm{i}\nu\rho}\hat{s}(\rho)F(\rho,\omega,\theta)+\mathrm{e}^{-\mathrm{i}\nu\rho}\hat{s}(-\rho)F(\rho,-\omega,-\theta)\,\mathrm{d}\rho$$
$$=\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^\infty \mathrm{e}^{\mathrm{i}\nu\rho}\hat{s}(\rho)F(\rho,\omega,\theta)\,\mathrm{d}\rho \tag{3.31}$$

where we have used the well known relation

$$\overline{F(\rho,\omega,\theta)}=F(\rho,-\omega,-\theta) \tag{3.32}$$

and we define $F(\rho,\omega,\theta)$ for $\rho<0$ as in (2.17), i.e.

$$F(\rho,\omega,\theta)=F(-\rho,-\omega,-\theta)\qquad \rho<0;\ \omega,\theta\in S^2. \tag{3.33}$$

Thus we obtain (2.18) in theorem 2.2, which completes the proof.

Let us finish this section with a remark on the complex-valued scattered wave $v^{\mathrm{sc}}$. In the proof of theorem 2.2 the complex-valued scattering wave $v^{\mathrm{sc}}$ was used only to introduce the real-valued scattering wave $u^{\mathrm{sc}}$. However, it might be possible to use $v^{\mathrm{sc}}$ for quantum mechanical problems where it is necessary for the probability amplitudes to be complex functions.

## 4. Inverse scattering problem

We have seen that, by launching the incident pulses $u_0(t,x,\theta,s)$ and evaluating the scattered wave $u^{\mathrm{sc}}(t,x,\theta,s)$, the asymptotic wavefunction $K(\nu,\omega,\theta,s)$ can be recovered. Through the asymptotic wavefunction $K(\nu,\omega,\theta,s)$ we shall be able to recover the scattering amplitude $F(\rho,\omega,\theta)$ associated with the Schrödinger operator $H=-\Delta+V(x)$. Wilcox (1983) used the Born approximation and the Radon transform to recover the coefficients of the acoustic wave equation. However, since we are now dealing with the Schrödinger operator, we have some other ways to recover the potential $V(x)$. One is Newton's method (Newton 1980) which approaches the problem by giving an integral equation to be solved. Another way is to use the high-energy method of Saitō (1982a, b). Let

$$\phi_{k,x}(\omega)=\exp(-\mathrm{i}kx\cdot\omega)\qquad \omega\in S^2. \tag{4.1}$$

We regard $\phi_{k,x}$ as a function defined on $S^2$ with parameters $k>0$ and $x\in\mathbb{R}^3$. Set

$$g(x,k)=k^2\int_{S^2}\int_{S^2}F(k,\omega,\omega')\phi_{k,x}(\omega)\overline{\phi_{k,x}(\omega')}\,\mathrm{d}\omega\,\mathrm{d}\omega' \tag{4.2}$$

where $F(k,\omega,\omega')$ is the scattering amplitude as above. Then the following theorem is the main result of Saitō (1982a, b).

*Theorem 4.1.* Let $V(x)$ satisfy assumption 2.1, let $\phi_{k,x}$ and $g(x,k)$ be as above. Then the limit

$$g(x,\infty)=\lim_{k\to\infty}g(x,k)=-2\pi\int_{\mathbb{R}^3}\frac{V(y)}{|x-y|^2}\,\mathrm{d}y \tag{4.3}$$

exists for all $x\in\mathbb{R}^3$. The potential $V(x)$ is recovered by the formula

$$V(x)=-\frac{1}{4\pi}\mathscr{F}_0^*(|\xi|\mathscr{F}_0 g)(x). \tag{4.4}$$

Theorem 4.1 is true even for a general short-range potential $V(x)$ satisfying

$$|V(x)| \leq C(1+|x|)^{-1-\varepsilon} \qquad x \in \mathbb{R}^3 \tag{4.5}$$

with positive constants $C$ and $\varepsilon$ (Saitō 1982a, b). In Saitō (1984), theorem 4.1 is also extended to the potential defined on $\mathbb{R}^n$, $n \geq 2$. Set

$$V(x, k) = -\frac{1}{4\pi}\mathscr{F}_0^*(|\xi|\mathscr{F}_0 g(\cdot, k))(x). \tag{4.6}$$

Then we have the following approximation formula (Saitō 1986).

*Theorem 4.2.* Let us assume that $V(x) \leq C(1+|x|)^{-2}$ with $C > 0$ and $\eta > 7/4$, and let us also assume that $V(x)$ is a $C^2$ function satisfying

$$|D^\alpha V(x)| \leq C'(1+|x|)^{-\beta} \qquad x \in \mathbb{R}^3;\ |\alpha| = 1, 2 \tag{4.7}$$

with constants $C' > 0$ and $\beta > \frac{5}{2}$, where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi-index, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$ and

$$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} D_3^{\alpha_3} \qquad D_j = \partial/\partial x_j;\ j = 1, 2, 3. \tag{4.8}$$

Then we have the estimate

$$\| V(\cdot) - V(\cdot, k)\|_{L^2(\mathbb{R}^3)} \leq C_1 k^{-1} \qquad k \to \infty \tag{4.9}$$

where the constant $C_1$ depend on $C'$, $\beta$ and $\max|V(x)|$, but does not depend on $k$.

Among other works on the inverse scattering for the plasma wave equations, DeFacio and Rose (1985) and Rose *et al* (1985) should be mentioned as a hybrid method of the above works. Thus we have shown that the results of the inverse scattering for the reduced wave equations can be used to solve the inverse scattering problem for the wave equations in the case of the plasma wave equation.

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